The decay of the free motion of a floating body

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A body floating on the free surface of water is given a small vertical displacement from its equilibrium position and is then held fixed. When the fluid has again come to rest the body is released. The subsequent damped motion is investigated when viscosity and surface tension are neglected and the equations of motion are linearized. The method applies to bodies of arbitrary shape in two or three dimensions, and is described in detail here for the heaving motion of a horizontal half-immersed circular cylinder of radius a.

The *forced periodic* motion of such a cylinder has been studied in earlier papers. In particular, the hydrodynamic forces exerted by the fluid on the body can be described by a dimensionless coefficient, $\Lambda(\omega(a/g)^{\frac{1}{2}})$, where ω is the (real) angular frequency. The function Λ can be found by convergent infinite processes, but not explicitly, and the difficulties of the problem are due to this. The free motion of the cylinder is solved in the present paper by Fourier methods. The motion is regarded as the superposition of simple harmonic motions, and the displacement $y_0(t)$ is thus obtained in the form of a Fourier integral

$$y_0(\tau(a/g)^{\frac{1}{2}}) = -\tfrac{1}{8}iy_0(0)\int_{-\infty}^{\infty}\frac{u(1+\Lambda(u))\,e^{-iu\tau}du}{1-\frac{1}{4}\pi u^2(1+\Lambda(u))}.$$

It is seen that the integrand involves the force coefficient Λ and is thus not strictly an explicit expression. The asymptotic behaviour for large times can be found explicitly when the depth is infinite:

$$y_0(t) \sim -\frac{4}{\pi}y_0(0)\frac{a}{gt^2}.$$

A damped harmonic behaviour had been expected. The slow monotonic decay occurs because the function $\Lambda(\omega(a/g)^{\frac{1}{2}})$, when continued into the complex ω-plane, can be shown to be many-valued near $\omega = 0$. No physical interpretation has yet been found for this property. The free motion of a cylinder set in motion by an applied force is also treated, with similar results.

Reasons are given why there are no rapidly oscillatory terms in the asymptotic expression. For finite constant depth the function $\Lambda(\omega(a/g)^{\frac{1}{2}})$ is single-valued near $\omega = 0$, and the asymptotic expression for this case is not yet known.

1. Introduction

A body floating on the free surface of water is given a small vertical displacement from its equilibrium position and is then held fixed. When the fluid has again come to rest, the body is released and a motion of the body and fluid ensues, subject only to the external force of gravity. The amplitude of motion of the body soon reaches a maximum and then decays as energy is progressively transferred away from the body into the fluid by waves and by viscous action. Ultimately the body and the fluid return to their equilibrium state of rest. The motion is damped by waves even when viscosity is negligible, and in the present paper we shall consider the motion of a body subject only to wave damping in a frictionless fluid under gravity. Surface tension is also neglected.

For comparison, let us consider the analogous problem in acoustics. A rigid body is surrounded by air (in which the velocity of sound is independent of the frequency) and is subject to an elastic restoring force. It is slightly displaced from its equilibrium position and released from rest. When the body is a sphere, the resulting acoustic wave motion and the motion of the body can both be found explicitly (Love 1904; Lamb 1932, § 301). The motion of the body is the superposition of a finite number of exponentially damped harmonic modes. In practice the damping of the modes is very light and can be calculated quite accurately from the energy radiated in one cycle of a forced *exactly periodic* motion, where the period is determined by the mass of the body (corrected for the virtual mass of the air) and the magnitude of the elastic restoring force. In our water-wave problem, on the other hand, the wave velocity depends on the wavelength, there are no known explicit solutions (not even for the forced periodic motion of a circular cylinder or a sphere), and the damping is not light.

To describe some earlier attempts at the solution of this problem it is first necessary to consider the form of the equations of motion. To fix ideas, let us confine attention to the heaving (i.e. vertical) motion of a three-dimensional body of such symmetry that heaving is independent of rolling and pitching. (Similar considerations also apply to less symmetrical bodies.) At any instant t the motion is described by the vertical displacement $y_0(t)$ of the body, and by the velocity potential $\phi(x, y, z; t)$ at any point of the fluid. These functions are coupled through boundary conditions on the body, which contain time derivatives and thus connect the motion at time t with the motion at neighbouring instants of time. Time derivatives also occur in the boundary condition satisfied by the potential at the free surface. In the exact non-linear problem the non-linear boundary conditions must be applied at the instantaneous position of the body and of the free surface, while in the linearized (small-amplitude) problem the boundary conditions are applied on fixed surfaces. We observe that three space dimensions and one time dimension are involved. In the present paper we shall be concerned only with the linearized problem which can be reduced in various ways to the solution of integro-differential equations in a smaller number of dimensions. One such reduction is due to Sretenskii (1937) whose work is described in detail by Wehausen & Laitone (1960, pp. 619–620). (I have not seen the original paper.) The fluid motion is represented by a distribution of instantaneous (Cauchy–Poisson) wave sources over the surface of the body. The unknown functions are now the displacement $y_0(t)$ and the instantaneous source strength at any point on the body at any time. Only 2 space dimensions are now involved which describe the surface of the body. The unknown functions are connected by a pair of complex linear integro-differential equations which appear too complicated for analytical treatment. For certain nearly vertical thin and slender bodies Sretenskii assumes a relation between body shape and source strength, and obtains a single integro-differential equation for the displacement which he solves numerically for a certain thin wedge (see Wehausen & Laitone, p. 620).

Another reduction to an integro-differential equation is due to Cummins (1962). Suppose that the Cauchy–Poisson type of wave motion due to a sudden initial displacement of the body can be found, the body being held fixed after the initial instant. (No solution of this type is known at present.) The force on the body at any time can then be found and gives the coefficient function in an integro-differential equation of simple type for the displacement. No numerical or analytical deductions from this equation are yet available.

The related but simpler problem of a flat body in very shallow water has been solved explicitly by John (1949). This is simpler because here (as in the acoustic problem described above) the wave velocity is independent of the wavelength.

In the work described in the present paper, as in the work described above, the fluid will be assumed inviscid and the equations will be linearized. The depth of the fluid will be assumed infinite; the method also applies to finite depth but the asymptotic results are different. We shall concentrate on the vertical displacement $y_0(t)$ of the body for which we shall obtain not an integro-differential equation but an explicit solution in integral form. The integrand involves, however, the *force coefficient*, a function of the frequency which describes the force acting on the body *in forced periodic* heaving. This force coefficient is not known explicitly in closed form but can be found for any desired frequency by solving an integral equation or an equivalent infinite system of linear equations. The method is applicable to bodies of general shape in two or three dimensions but for the sake of simplicity will be applied here in detail to one specific two-dimensional problem, the free heaving of a half-immersed horizontal circular cylinder. This is chosen because for the forced periodic heaving of such a cylinder we already possess analytical and numerical results. In particular, the force coefficient $\Lambda(\omega(a/g)^{\frac{1}{2}})$ at any given angular frequency ω may be regarded as known, though not in closed form (see § 3 and the appendix below).

We shall express the free heaving motion by Fourier's theorem as the superposition of periodic motions. For each frequency component the fluid action on the body is completely expressed by the above-mentioned function. In this way the displacement is expressed as a Fourier integral involving Λ. This integral can be computed but in the present paper we shall be concerned mainly with the asymptotic behaviour of $y_0(t)$ for large values of t, which can be obtained explicitly because $\omega = 0$ is a logarithmic branch point of the function Λ. It will be shown that $y_0(t)$ is ultimately non-oscillatory. It is believed that these results are applicable to bodies of general shape in 2 and 3 dimensions if the depth is infinite, but that the asymptotic treatment for finite depth will be more difficult. We shall first write down the equations of motion, next consider the related problem of a cylinder set in motion by an applied force, and then consider the problem posed at the beginning of the introduction.

2. The equations of motion

It is assumed that the equilibrium position of the centre of the circular cylinder is in the mean free surface. This point is taken as the origin of rectangular Cartesian co-ordinates (x, y). The x-axis is horizontal and y increases with depth. Polar co-ordinates (r, θ) are defined by $x = r\sin\theta$, $y = r\cos\theta$; thus the equilibrium position of the cylinder is $r = a$, where a is the radius of the cylinder. The vertical displacement $y_0(t)$ of the cylinder is to be found. The amplitude of motion is assumed to be so small that all equations can be linearized. Since viscosity is neglected and the density ρ of the fluid is constant, it is possible to describe the motion of the fluid by a velocity potential $\phi(x, y;\, t)$ satisfying the equation of continuity

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi(x,y;t) = 0 \quad \text{in the region} \quad r > a, \quad y > 0, \tag{2.1}$$

The linearized condition of constant pressure at the free surface is

$$\frac{\partial^2\phi}{\partial t^2} - g\frac{\partial\phi}{\partial y} = 0 \quad \text{when} \quad y = 0, \quad |x| > a; \tag{2.2}$$

cf. Lamb (1932, §227). On the cylinder the radial velocity components of the body and of the fluid are equal,

$$\frac{\partial\phi}{\partial r} = \dot{y}_0(t)\cos\theta \quad \text{when} \quad r = a, \quad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi. \tag{2.3}$$

Finally, there is the equation of motion of the body

$$\tfrac{1}{2}\pi\rho a^2\ddot{y}_0(t) = -2\rho g a y_0(t) + 2\rho a\int_0^{\frac{1}{2}\pi}\cos\theta\frac{\partial}{\partial t}\phi(a\sin\theta, a\cos\theta;\, t)\,d\theta + f_0(t), \tag{2.4}$$

where on the right-hand side the first term is the hydrostatic restoring force, the second term is the resultant of the hydrodynamic pressures, and the third term is the applied vertical force. The mass of the body is $\frac{1}{2}\pi\rho a^2$ (per unit width), by the principle of Archimedes. As was explained in §1 above, we shall use the Fourier transforms of these equations. We suppose that there is no motion when $t < 0$ and write

$$\Phi(x, y;\, \omega) = \int_0^\infty e^{i\omega t}\phi(x, y;\, t)\,dt, \tag{2.5}$$

$$Y_0(\omega) = \int_0^\infty e^{i\omega t}y_0(t)\,dt, \tag{2.6}$$

$$F_0(\omega) = \int_0^\infty e^{i\omega t}f_0(t)\,dt.$$

If the total energy of motion is finite, the potential energy of the body remains bounded, and therefore $y_0(t)$ is bounded. It follows that the function $Y_0(\omega)$ of the complex frequency ω is regular in the whole of the upper half ω-plane $\mathscr{I}\omega > 0$. If we make the trivial transformation $\omega = is$ then (2.5) and (2.6) are Laplace transforms which are widely used in initial-value problems. The inversion formulae are

$$\phi(x,y;\,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\,\Phi(x,y;\,\omega)\,d\omega, \tag{2.7}$$

$$y_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}Y_0(\omega)\,d\omega. \tag{2.8}$$

Integration by parts gives

$$\int_0^\infty e^{i\omega t}\dot{y}_0(t)\,dt = -i\omega\int_0^\infty e^{i\omega t}y_0(t)\,dt - y_0(0)$$
$$= -i\omega Y_0(\omega) - y_0(0), \tag{2.9}$$

$$\int_0^\infty e^{i\omega t}\ddot{y}_0(t)\,dt = -i\omega\int_0^\infty e^{i\omega t}\dot{y}_0(t)\,dt - \dot{y}_0(0)$$
$$= -\omega^2 Y_0(\omega) + i\omega y_0(0) - \dot{y}_0(0), \tag{2.10}$$

and similar expressions involving the potential $\phi(x,y;\,t)$. From these the Fourier transforms of the equations of motion are readily obtained; see §§ 3 and 4 below.

3. Cylinder set in motion by an applied force

In this section we shall suppose that a prescribed vertical force $f_0(t)$ acts for a finite time, and that thereafter the system is left to itself. We impose the initial conditions $y_0(0) = \dot{y}_0(0) = 0$. On the free surface ($y = 0$, $|x| > a$), we have initially $\phi = 0$ (zero impulsive pressure), and $\dot{\phi} = 0$ (zero elevation). Now let the Fourier operator $\int_0^\infty e^{i\omega t}\ldots dt$ be applied to the equations of motion (2.1) to (2.4). On taking account of initial conditions we find that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi(x,y;\,\omega) = 0 \quad\text{when}\quad r > a,\quad y > 0; \tag{3.1}$$

$$\left(\omega^2 + g\frac{\partial}{\partial y}\right)\Phi(x,y;\,\omega) = 0 \quad\text{when}\quad y = 0,\quad |x| > a; \tag{3.2}$$

$$\frac{\partial\Phi}{\partial r} = -i\omega Y_0(\omega)\cos\theta \quad\text{when}\quad r = a,\quad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi; \tag{3.3}$$

$$-\tfrac{1}{2}\pi\rho a^2\omega^2 Y_0(\omega) = -2\rho g a Y_0(\omega) - 2\rho a i\omega\int_0^{\frac{1}{2}\pi}\Phi(a\sin\theta, a\cos\theta;\,\omega)\cos\theta\,d\theta + F_0(\omega). \tag{3.4}$$

Equations (3.1) to (3.3) are evidently identical, for real ω, with equations describing the fluid motion due to the forced periodic heaving (of constant amplitude $Y_0(\omega)$) of the circular cylinder (see Ursell 1949). To define Φ completely an additional radiation condition at infinity is needed. If ω is a complex frequency in the upper half plane, let it be assumed that $\Phi(x,y;\,\omega)$ defined by (2.5) does not tend to infinity as x and y tend to infinity. (Compare the discussion following equation (2.6) above.) It can then be shown that at large distances the potential is the sum of progressive outward-travelling waves

$$\frac{\partial\Phi}{\partial r} \mp \frac{i\omega^2}{g}\Phi \to 0 \quad\text{as}\quad r\to\infty, \tag{3.5}$$

according as $\omega \gtrless 0$. (The same condition can be obtained by use of the Rayleigh friction; cf. Lamb 1932, § 242). For positive ω this radiation condition is also the same as for the forced periodic heaving of the cylinder, and we conclude that $\Phi(x, y;\, \omega)$ is proportional to the potential obtained by Ursell (1949) for positive ω, and to its complex conjugate for negative ω. Clearly the constant of proportionality contains $Y_0(\omega)$ as a factor. If the function $\Phi(x, y;\, \omega)$ is now regarded as known, though not explicitly in closed form, the equation

$$\int_0^{\frac{1}{2}\pi} \Phi(a\sin\theta, a\cos\theta;\, \omega)\cos\theta\, d\theta = \tfrac{1}{4}\pi i a\omega Y_0(\omega)\, \Lambda_1(\omega, a, g) \tag{3.6}$$

defines a non-dimensional function $\Lambda_1(\omega, a, g)$ which is in fact a function of the single variable $\omega(a/g)^{\frac{1}{2}}$,

$$\Lambda_1(\omega, a, g) = \Lambda(\omega(a/g)^{\frac{1}{2}}), \tag{3.7}$$

as is obvious from dimensional reasoning. The function is in principle a known function which can be deduced from published computations on periodic heaving (Ursell 1957). The real part of $\Lambda(\omega(a/g)^{\frac{1}{2}})$ is the virtual mass coefficient, the imaginary part is simply related to the wave-making coefficient. In fact, if $A(\omega)$ is the wave amplitude at infinity, then the imaginary part of $\Lambda(\omega(a/g)^{\frac{1}{2}})$ is $\pm\, 2g^2\pi^{-1}a^{-2}\omega^{-4}(A(\omega)/Y_0(\omega))^2$ according as $\omega \gtrless 0$. (It is shown in the appendix below that Λ has a logarithmic branch point at $\omega = 0$.) It now follows from (3.4), (3.6) and (3.7) that

$$Y_0(\omega) = \frac{1}{2\rho g a}\, \frac{F_0(\omega)}{1 - (\pi a\omega^2/4g)\,(1 + \Lambda(\omega(a/g)^{\frac{1}{2}}))}, \tag{3.8}$$

whence, from (2.8),

$$y_0(t) = \frac{1}{4\pi\rho g a}\int_{-\infty}^{\infty} \frac{F_0(\omega)\, e^{-i\omega t}\, d\omega}{1 - (\pi a\omega^2/4g)\,(1 + \Lambda(\omega(a/g)^{\frac{1}{2}}))}, \tag{3.9}$$

$$= \frac{1}{2\rho g^{\frac{1}{2}} a^{\frac{3}{2}}}\int_0^t f_0(t')\, h_1\!\left((t - t')\left(\frac{g}{a}\right)^{\frac{1}{2}}\right) dt' \tag{3.10}$$

by the convolution theorem (Titchmarsh 1948, p. 59). The function $f_0(t)$ is the applied vertical force, the function h_1 is defined by the equation

$$h_1(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-iu\tau}\, du}{1 - \frac{1}{4}\pi u^2\,(1 + \Lambda(u))}, \tag{3.11}$$

and vanishes for $\tau \leqslant 0$. In (3.11) the function $\Lambda(u)$ is the complex-valued force coefficient described above. The expression (3.10) shows that the displacement $y_0(t)$ is a linear convolution transform of the force $f_0(t)$, as might have been expected. We have assumed that the real and imaginary parts of the denominator do not both vanish at the same point on the real ω-axis; if they did so, then the cylinder could oscillate freely at that frequency without damping. Published computations (Ursell 1957) show that for the heaving circular cylinder there is in fact damping at all frequencies. (The energy considerations of § 2 showed, moreover, that for motions of finite energy the function $Y_0(\omega)$ is regular in the upper half ω-plane.)

Equations (3.9) and (3.10) solve our problem in a form involving convergent integrals. For any given $f_0(t)$ they can be computed, but direct quadrature is not convenient for large values of t when the integrand oscillates rapidly. Numerical results are not yet available but it is remarkable that the dominant asymptotic behaviour for large t can be found analytically, although $\Lambda(u)$ is not known in closed form. The behaviour of Fourier integrals like (3.9) and (3.11) for large t is found as follows: let the variable ω (or $u = \omega(a/g)^{\frac{1}{2}}$) be regarded as a complex variable, and the integral as a contour integral. In general, the integrand has singularities in the lower half ω-plane. It is then known (see e.g. Carslaw & Jaeger 1947, p. 279) that the asymptotic behaviour is dominated by the behaviour of the integrand at those singular points which lie on or nearest to the real ω-axis. In our problem these are (i) the logarithmic branch point at $\omega = 0$ due to the logarithmic branch point of the function Λ; (ii) possibly the points $\omega = \pm\infty$ on the real ω-axis if Λ is sufficiently singular there.

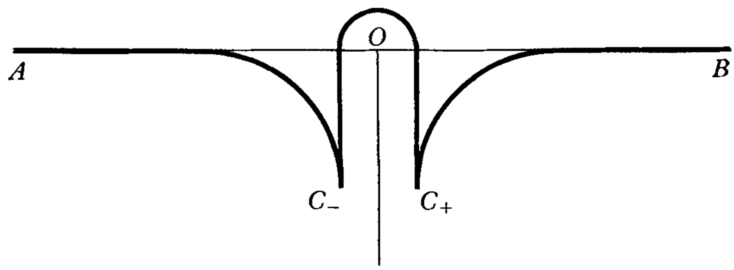

FIGURE 1. The path of integration in the u-plane.

The contour of integration AOB of (3.11) along the real ω-axis is therefore deformed into the contour $AC_-\,OC_+\,B$ where AC_- and C_+B coincide with the real axis except near $\omega = 0$. The points C_-, C_+ lie on opposite sides of the negative imaginary ω-axis which is a branch cut. The contour is chosen so that there are no singularities above it (see figure 1). The value of the integral is clearly unchanged by the deformation of the contour. The following argument is given for the function $h_1(\tau)$ defined by (3.11) but applies with little change to (3.9).

The contribution from $C_-\,OC_+$ is found from Watson's lemma (Erdélyi 1956, p. 31), for which the leading terms in the expansion of the integrand near $u = 0$ are needed. It is clear that single-valued terms make no contribution. From (A 1.18) in the appendix we see that, along OC_+ and OC_-,

$$
\begin{aligned}
&[1-\tfrac{1}{4}\pi u^2(1+\Lambda(u))]^{-1}\\
&\quad= \left[1+\tfrac{1}{4}\pi|u|^2+\tfrac{1}{2}\pi|u|^2\frac{\tfrac{3}{2}-2\ln 2-\gamma-2(\ln|u|\mp i\pi)+\ldots}{1+\tfrac{1}{4}\pi|u|^2(\ln|u|\mp i\pi)+\ldots}\right]^{-1}\\
&\quad= 1-\tfrac{1}{4}\pi|u|^2-\tfrac{1}{2}\pi|u|^2\{\tfrac{3}{2}-2\ln 2-\gamma-2(\ln|u|\mp i\pi)\}+\text{smaller terms},
\end{aligned}
\tag{3.12}
$$

whence

$$
\begin{aligned}
&\frac{1}{2\pi}\left(\int_{OC_+}-\int_{OC_-}\right)\frac{e^{-iu\tau}\,du}{1-\tfrac{1}{4}\pi u^2(1+\Lambda(u))}\\
&\quad= -\frac{i}{2\pi}\int_{OC_+} e^{-|u|\tau}\{\tfrac{1}{4}\pi|u|^2(\ln|u|-i\pi)-\tfrac{1}{4}\pi|u|^2(\ln|u|+i\pi)\}\,d|u|+\text{smaller terms}\\
&\quad\sim -\frac{4}{\pi}\int_0^\infty e^{-|u|\tau}|u|^2\,d|u| \sim -\frac{8}{\pi\tau^3}.
\end{aligned}
\tag{3.13}
$$

The contributions to $h_1(\tau)$ from AC_- and from C_+B are more difficult to estimate; the method of repeated integration by parts (Erdélyi 1956, p. 47) succeeds only if the first few derivatives of $\Lambda(u)$ do not oscillate rapidly at infinity. If they do oscillate, then $h_1(\tau)$ may oscillate rapidly for large τ. There are, however, reasons for believing that the contributions from AC_- and C_+B are negligible, see § 5 below. We conclude from (3.13) that

$$h_1(\tau) \sim -\frac{8}{\pi\tau^3} \quad \text{as} \quad \tau \to \infty. \tag{3.14}$$

If the applied force acts for only a finite time, then clearly from (3.10)

$$\begin{aligned} y_0(t) &\sim -\frac{8a^{\frac{3}{2}}}{\pi t^3 g^{\frac{3}{2}}} \frac{1}{2\rho g^{\frac{1}{2}} a^{\frac{3}{2}}} \int_0^\infty f_0(t')\,dt' \\ &= -\frac{4}{\pi\rho g^2 t^3} \int_0^\infty f_0(t')\,dt', \end{aligned} \tag{3.15}$$

where in fact the integration extends only over a finite range. It is seen that $y_0(t)$ is ultimately non-oscillatory. Presumably the mean level of the fluid near the body is also given by (3.15). It is remarkable that a downward force causes a rise in mean level which decays only slowly.

4. Cylinder slightly displaced from equilibrium

This is the problem described in § 1 above. There is no applied force, the initial displacement $y_0(0)$ is prescribed, and the cylinder is released at time $t = 0$ with zero velocity. The initial velocity of the fluid is assumed to vanish. The procedure is similar to that of § 3, except that some of the initial-value terms in (2.9) and (2.10) must now be included.

The transformed equations are

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi(x,y;\,\omega) = 0 \quad \text{when} \quad r > a, \quad y > 0; \tag{4.1}$$

$$\left(\omega^2 + g\frac{\partial}{\partial y}\right)\Phi(x,y;\,\omega) = 0 \quad \text{when} \quad y = 0, \quad |x| > a; \tag{4.2}$$

$$\frac{\partial\Phi}{\partial r} = (-i\omega Y_0(\omega) - y_0(0))\cos\theta \quad \text{when} \quad r = a, \quad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi; \tag{4.3}$$

$$\begin{aligned} &\tfrac{1}{2}\pi\rho a^2\{-\omega^2 Y_0(\omega) + i\omega y_0(0)\} \\ &\quad = -2\rho g a Y_0(\omega) - 2\rho a i\omega \int_0^{\frac{1}{2}\pi} \Phi(a\sin\theta, a\cos\theta;\,\omega)\cos\theta\,d\theta; \end{aligned} \tag{4.4}$$

together with the radiation condition (3.5). Equations (4.1) to (4.3) are identical with equations (3.1) to (3.3), except for the coefficient in (4.3), and it is thus obvious that

$$\int_0^{\frac{1}{2}\pi} \Phi(a\sin\theta, a\cos\theta;\,\omega)\cos\theta\,d\theta = \tfrac{1}{4}\pi a(i\omega Y_0(\omega) + y_0(0))\,\Lambda(\omega\,(a/g)^{\frac{1}{2}}), \tag{4.5}$$

where Λ is the function defined by (3.6) and (3.7). On substituting (4.5) in (4.4) it is seen that

$$2\rho g a Y_0(\omega)\,\{1 - (\pi a\omega^2/4g)\,(1+\Lambda)\} = -\tfrac{1}{2}\pi\rho a^2 y_0(0)\,(i\omega + i\omega\Lambda), \tag{4.6}$$

whence
$$y_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y_0(\omega)\, e^{-i\omega t} d\omega$$
$$= -\tfrac{1}{8}i\frac{a}{g}y_0(0)\int_{-\infty}^{\infty} \frac{\omega\{1+\Lambda(\omega(a/g)^{\frac{1}{2}})\}\, e^{-i\omega t}\, d\omega}{1-(\pi a\omega^2/4g)\{1+\Lambda\}} \tag{4.7}$$
$$= -\tfrac{1}{8}iy_0(0)\int_{-\infty}^{\infty} \frac{u(1+\Lambda(u))\, e^{-iu\tau}\, du}{1-\frac{1}{4}\pi u^2(1+\Lambda(u))} \tag{4.8}$$
where $\tau = t(g/a)^{\frac{1}{2}}$. Thus
$$y_0(t) = y_0(0)\, h_2(t(g/a)^{\frac{1}{2}}) \tag{4.9}$$
where by definition
$$h_2(\tau) = -\tfrac{1}{8}i\int_{-\infty}^{\infty} \frac{u(1+\Lambda)\, e^{-iu\tau}\, du}{1-\frac{1}{4}\pi u^2(1+\Lambda)}. \tag{4.10}$$

We note that this is related to the function $h_1(\tau)$ defined by (3.11), since, when $\tau > 0$,
$$h_1(\tau)+h_2'(\tau) = \int_{-\infty}^{\infty} \frac{e^{-iu\tau}}{1-\frac{1}{4}\pi u^2(1+\Lambda)}\left(\frac{1}{2\pi}-\tfrac{1}{8}u^2(1+\Lambda)\right) du$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iu\tau}\, du = 0. \tag{4.11}$$

Also, from (4.9), $h_2(\tau) \to 1$ as $\tau \to 0$ through positive values. The asymptotic behaviour of $h_2(\tau)$ for large τ can be found by the method of § 3; alternatively, it may be inferred from (4.11). For
$$h_2(\tau) = \int_{\tau}^{\infty} h_1(\tau')\, d\tau' \tag{4.12}$$
$$\sim \int_{\tau}^{\infty}\left(-\frac{8}{\pi(\tau')^3}\right) d\tau' \quad \text{from (3.14)}$$
$$= -\frac{4}{\pi\tau^2}. \tag{4.13}$$
It follows from (4.9) that
$$y_0(t) \sim -\frac{4}{\pi}y_0(0)\frac{a}{gt^2}. \tag{4.14}$$

The remarks at the end of § 3 apply equally to the present problem, in which however the monotonic decay is even slower than before.

5. Discussion

In §§ 3 and 4 we have solved two initial-value problems by means of Fourier integrals. An explicit asymptotic treatment for large t was analytically possible because the origin $\omega = 0$ in the frequency plane is a logarithmic branch point, and gave the following results: a body initially depressed and then released from rest ultimately approaches its equilibrium position from above, the distance decaying like t^{-2} (see (4.14)). For a body set in motion by an applied force the results are similar, except that the decay varies like t^{-3} (see (3.15)). In this final stage any damped harmonic terms arising from complex zeros of the denominator of (4.8) are insignificant; this was not expected. I have not yet succeeded in finding a physical interpretation for the many-valued behaviour of the force-coefficient $\Lambda(u) = \Lambda(\omega(a/g)^{\frac{1}{2}})$ near $\omega = 0$ in the complex frequency plane.

In obtaining these results it was necessary to assume that the force coefficient $\Lambda(u)$ and its first few derivatives do not oscillate rapidly for large real values of u; otherwise rapidly oscillating terms like $\cos(gt^2/4a)$ might also appear in the asymptotic expressions. (For comparison, if a water surface is given a slight initial parabolic deformation in the range $-a < x < a$, then the surface velocity at $x = 0$ does contain such terms.) There remains the problem, not yet completely solved, of finding out whether our assumption is correct.

This can be expressed more precisely. We already know that, for real u,

$$\Lambda(u) \sim 1 - \frac{4}{3\pi u^2} + \ldots, \tag{5.1}$$

as was shown in an earlier paper (Ursell 1953, equation (5.1)), but now we need the behaviour in more detail. It is sufficient to show that (5.1) holds uniformly in a finite sector $-\epsilon \leqslant \arg u \leqslant 0$ of the complex u-plane, because then the curves AC_- and C_+B may be deformed into the lower half plane so as to make an angle with the real u-axis, and the contributions from large u are then evidently exponentially small. On re-examining the integral equations in earlier papers (particularly in Ursell 1961) it is found that in fact only a few changes are needed to extend (5.1) from the real axis into an angle $0 \leqslant \arg u \leqslant \epsilon$ but not into an angle $-\epsilon \leqslant \arg u \leqslant 0$. It is, nevertheless, believed that the result (5.1) is valid there also; arguments are given in the appendix below. It follows (cf. Erdélyi 1956, p. 21) that (5.1) may be differentiated any number of times, so that the derivatives of $\Lambda(u)$ do not oscillate rapidly for large u.

The analysis given in §§ 3 and 4 above is equally applicable to finite depth, if $\Lambda(u)$ is now interpreted as the force coefficient for finite depth. But the conclusions are different, for it appears that $\Lambda(u)$ is now single-valued near $u = 0$, and it becomes more difficult to locate the singularities nearest to the real u-axis. It seems likely that the motion is ultimately a damped harmonic oscillation, with a period depending on the depth rather than on the radius.

Appendix. The force coefficient $\Lambda(u)$ near $u = 0$ and $u = \infty$

The potential $\Phi(x, y;\, \omega)$ describes the forced heaving of a circular cylinder and was calculated in earlier papers (Ursell 1949, 1953, 1957). Here we are concerned with the derived quantity $\Lambda(u) = \Lambda(\omega(a/g)^{\frac{1}{2}})$ defined by (3.6) and (3.7).

(1) *The analytic form of* Λ *for small* u. This is needed in (3.12) above. The infinite processes used here can be rigorously justified, e.g. by the theory of infinite determinants.

The potential $\Phi(x, y;\, \omega)$ is expanded in the form (cf. Ursell 1949)

$$\frac{1}{a^2}\Phi = D(Ka)\,\Phi_0(Kr,\theta) + \sum_1^\infty \frac{\alpha_n(Ka)}{2n} a^{2n}\left(\frac{\cos 2n\theta}{r^{2n}} + \frac{K}{2n-1}\frac{\cos(2n-1)\theta}{r^{2n-1}}\right), \tag{A 1.1}$$

where $K = \omega^2/g$, and where it is first supposed (to fix ideas) that $\omega > 0$. The function Φ_0 is the source potential

$$\Phi_0(Kr,\theta) = \int_0^\infty e^{-ky}\frac{\cos kx}{k-K}\,dk = \int_0^\infty e^{-kr\cos\theta}\frac{\cos(kr\sin\theta)}{k-K}\,dk, \tag{A 1.2}$$

where the path of integration passes below the point $k = K$ on the real axis, so as to make Φ_0 satisfy the radiation condition (3.5). All the terms on the right-hand side of (A 1.1) are harmonic and satisfy the free-surface condition (3.2). The normal-velocity condition† (3.3)

$$\left\langle \frac{\partial}{\partial r}\Phi(Kr,\theta)\right\rangle = -i\omega Y_0(\omega)\cos\theta \tag{A 1.3}$$

is also satisfied if the equation

$$D(Ka)\left\langle a\frac{\partial}{\partial r}\Phi_0(Kr,\theta)\right\rangle - \sum_1^\infty \alpha_n(Ka)\left(\cos 2n\theta + \frac{Ka}{2n}\cos(2n-1)\theta\right) = -i\omega a^{-1}Y_0(\omega)\cos\theta \tag{A 1.4}$$

holds in the range $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. This condition determines the coefficients $D(Ka), \alpha_n(Ka)$.

It is convenient to begin by eliminating $Y_0(\omega)$. Integration of (A 1.4) over $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ gives

$$D(Ka)\int_0^{\frac{1}{2}\pi}\left\langle a\frac{\partial}{\partial r}\Phi_0(Kr,\theta')\right\rangle d\theta' - \sum_1^\infty \alpha_n(Ka)\frac{(-1)^{n-1}Ka}{2n(2n-1)} = -i\omega a^{-1}Y_0(\omega). \tag{A 1.5}$$

(A 1.4) and (A 1.5) together give

$$D(Ka)\left[\left\langle a\frac{\partial}{\partial r}\Phi_0(Kr,\theta)\right\rangle - \cos\theta\int_0^{\frac{1}{2}\pi}\left\langle a\frac{\partial}{\partial r}\Phi_0(Kr,\theta')\right\rangle d\theta'\right] = \sum_1^\infty \alpha_n(Ka)\left[\cos 2n\theta + \frac{Ka}{2n}\left(\cos(2n-1)\theta - \frac{(-1)^{n-1}}{2n-1}\cos\theta\right)\right]. \tag{A 1.6}$$

The functions in brackets on the right-hand side involve Ka as a linear factor only. The function on the left-hand side involves terms like $(Ka)^m$ and

$$(Ka)^m(\ln Ka - i\pi),$$

as is readily seen from the expansion for Φ_0

$$\Phi_0(Kr,\theta) = -\sum_0^\infty(-1)^m(\ln Kr - i\pi)\frac{(Kr)^m\cos m\theta}{\Gamma(m+1)} + \sum_0^\infty(-1)^m\frac{\psi(m+1)}{\Gamma(m+1)}(Kr)^m\cos m\theta + \sum_1^\infty(-1)^m\frac{(Kr)^m\theta\sin m\theta}{\Gamma(m+1)}, \tag{A 1.7}$$

where $\psi(z) = d\ln\Gamma(z)/dz$ (Erdélyi 1953, vol. 1, p. 15). This can be obtained by considering that the right-hand side of (A 1.7) is a harmonic function which is an even function of θ, and which for $\theta = 0$ coincides with the expansion for

$$-e^{Kr}E_1(Kre^{i\pi}) = \oint_0^\infty e^{-kr}\frac{dk}{k-K} = \Phi_0(Kr,0);$$

here E_1 is an exponential integral, (Erdélyi 1953, vol. 2, p. 143). It follows from (A 1.7) that

$$\left\langle a\frac{\partial}{\partial r}\Phi_0(Kr,\theta)\right\rangle = -1 - \sum_1^\infty(-1)^m(\ln Ka - i\pi)\frac{(Ka)^m}{\Gamma(m)}\cos m\theta + \sum_1^\infty(-1)^m\frac{\psi(m)}{\Gamma(m)}(Ka)^m\cos m\theta + \sum_1^\infty(-1)^m\frac{(Ka)^m\theta\sin m\theta}{\Gamma(m)}. \tag{A 1.8}$$

† Angular brackets $\langle\ \rangle$ are used to indicate that r is to be put equal to a.

Now for small values of Ka the series on the right-hand side of (A 1.6) resembles a Fourier cosine series. Let therefore the integral operators

$$\int_0^{\frac{1}{2}\pi} \cos 2n\theta \ldots d\theta \quad (n = 0, 1, 2, \ldots)$$

be applied to (A 1.6). An infinite system of equations is thus obtained for the infinite number of unknowns $\alpha_n(Ka)$. This may be solved by iteration (Ursell 1949, p. 226), or by the theory of infinite determinants. From the analytic form of the various terms in (A 1.6) it then follows that

$$\frac{\alpha_n(Ka)}{D(Ka)} = (\ln Ka - i\pi)\,\alpha_n^*(Ka) + \alpha_n^{**}(Ka), \tag{A 1.9}$$

where α_n^*, α_n^{**} are power series in Ka with real coefficients, convergent for small Ka. Substitution in (A 1.5) gives

$$i\omega a^{-1} Y_0(\omega) = D(Ka)\,\{(\ln Ka - i\pi)\, Y_0^*(Ka) + Y_0^{**}(Ka)\}, \tag{A 1.10}$$

and substitution in (A 1.1) gives

$$\frac{1}{a^2}\int_0^{\frac{1}{2}\pi} \langle \Phi \rangle \cos\theta\, d\theta = D(Ka)\,\{(\ln Ka - i\pi)\, Z_0^*(Ka) + Z_0^{**}(Ka)\}, \tag{A 1.11}$$

where Y_0^*, Y_0^{**}, Z_0^*, Z_0^{**} are real power series convergent for small Ka. Thus, with a change of notation, we see from (A 1.10) and (A 1.11) that the force coefficient is given by

$$\Lambda = \frac{4\int_0^{\frac{1}{2}\pi} \langle \Phi \rangle \cos\theta\, d\theta}{\pi i a \omega Y_0(\omega)} = \frac{(\ln Ka - i\pi)\, A_1^*(Ka) + A_1^{**}(Ka)}{(\ln Ka - i\pi)\, A_2^*(Ka) + A_2^{**}(Ka)}, \tag{A 1.12}$$

where the A's are real power series, convergent for small Ka. We have so far assumed that $\omega > 0$. When $\omega < 0$, then $\ln Ka - i\pi$ must be replaced by $\ln Ka + i\pi$. It follows from this statement, since $Ka = u^2 = \omega^2 a g^{-1}$, that near $u = 0$ the function Λ is single-valued in the u-plane cut along the negative imaginary u-axis.

For our purpose we need the leading terms in the A's. If we wish to retain only the leading terms on the left-hand side of (A 1.6), we have from (A 1.8)

$$\left\langle a\frac{\partial}{\partial r}\Phi_0 \right\rangle = -1 + (\ln Ka - i\pi - \psi(1))\, Ka\cos\theta - Ka\,\theta\sin\theta + O((\ln Ka - i\pi)\,(Ka)^2), \tag{A 1.13}$$

whence

$$\left\langle a\frac{\partial}{\partial r}\Phi_0 \right\rangle - \cos\theta \int_0^{\frac{1}{2}\pi} \left\langle a\frac{\partial}{\partial r}\Phi_0 \right\rangle d\theta' = -1 + \tfrac{1}{2}\pi\cos\theta + O(Ka). \tag{A 1.14}$$

On following the Fourier procedure described above, we see that

$$\begin{aligned} \frac{\alpha_n(Ka)}{D(Ka)} &= \frac{4}{\pi}\int_0^{\frac{1}{2}\pi} (-1 + \tfrac{1}{2}\pi\cos\theta)\cos 2n\theta\, d\theta + O(Ka) \\ &= (-1)^{n-1}\frac{2}{4n^2 - 1} + O(Ka), \end{aligned} \tag{A 1.15}$$

whence, from (A 1.5),

$$-i\omega a^{-1}\frac{Y_0(\omega)}{D(Ka)} = -\tfrac{1}{2}\pi + (\ln Ka - i\pi) + O(Ka).$$

From (A 1.15) and (A 1.1) we find that

$$\frac{\int_0^{\frac{1}{2}\pi}\langle\Phi\rangle\cos\theta\,d\theta}{a^2D(Ka)} = -(\ln Ka - i\pi) - \gamma + \sum_1^\infty \frac{1}{n(4n^2-1)^2} + O(Ka)$$
$$= -(\ln Ka - i\pi) - \gamma + \tfrac{3}{2} - 2\ln 2 + O(Ka),$$

on summing the series. Hence, when $\omega > 0$,

$$\Lambda\,(\omega(a/g)^{\frac{1}{2}}) = \frac{4\int_0^{\frac{1}{2}\pi}\langle\Phi\rangle\cos\theta\,d\theta}{\pi i a\omega Y_0(\omega)}$$
$$= \frac{8}{\pi^2}\frac{-(\ln Ka - i\pi) - \gamma + \frac{3}{2} - 2\ln 2 + \ldots}{1-(2/\pi)Ka(\ln Ka - i\pi) + \ldots}; \tag{A 1.16}$$

i.e. since $Ka = u^2$,

$$\Lambda(u) = \frac{8}{\pi^2}\frac{-2(\ln u - \frac{1}{2}i\pi) + \frac{3}{2} - 2\ln 2 - \gamma + \ldots}{1-(4/\pi)\,u^2\,(\ln u - \frac{1}{2}i\pi) + \ldots}. \tag{A 1.17}$$

It follows that along OC_+ ($\arg u = -\frac{1}{2}\pi$) and OC_- ($\arg u = \frac{3}{2}\pi$) we have

$$\Lambda(u) = \frac{8}{\pi^2}\frac{-2(\ln|u| \mp i\pi) + \frac{3}{2} - 2\ln 2 - \gamma + \ldots}{1+(4/\pi)\,|u|^2\,(\ln|u| \mp i\pi) + \ldots}, \tag{A 1.18}$$

respectively. The approximations (A 1.18) are sufficient for our needs.

By a refinement of the preceding argument, analogous to Fredholm's theory of integral equations, it can be shown that $\Lambda(u)$ is an analytic function of u in the entire complex plane cut along the negative imaginary axis, and that it can be represented by an expression of the form (A 1.12), where the A's are now series with real coefficients convergent for all Ka.

(2) *Asymptotic behaviour for large complex u.* (See the discussion following equation (5.1) above.) The behaviour for large real u was considered in an earlier paper (Ursell 1953); equations quoted from that paper are distinguished by the letter O. We examine here how the methods of that paper can be adapted when $-\epsilon \leqslant \arg u \leqslant \epsilon$.

The variable $N = u^2$ was used in the earlier paper. An integral equation (O 3.15) of the form

$$\phi(\alpha; N) + \int_0^{\frac{1}{2}\pi}\phi(\theta;\, N)\,\mathfrak{K}(\theta,\alpha;\, N)\,d\theta = \text{known function of } \alpha \text{ and } N$$

was chosen for the values $\phi(\theta;\, N)$ of the potential on the circle, in such a way that the kernel $\mathfrak{K}(\theta,\, \alpha;\, N)$ tends to zero as N tends to infinity; there are infinitely many such *small kernels* $\mathfrak{K}$. The equation was then solved by iteration, and the asymptotic form of the force coefficient for real N was obtained (O 5.1, O 5.7)

$$\Lambda(u) \sim 1 - \frac{4}{3\pi u^2} + \ldots. \tag{A 2.1}$$

The procedure for constructing the small kernel $\mathfrak{K}$ was as follows. The kernel derived from a simple wave source (see O 3.3) contains large wave terms which were almost entirely removed by placing a wave source of suitable strength (depending on α) at the centre of the circle. The wave terms can in fact be cancelled exactly (cf. Ursell 1961, p. 644) and we shall now suppose this done. The kernel of the resulting equation (cf. O 3.7, O 3.14) is then of the form

$$\mathfrak{K}(\theta,\alpha;\ N) = \mathfrak{K}^{(1)}(\theta,\alpha;\ N)+e^{iN}\mathfrak{K}^{(2)}(\theta,\alpha;\ N), \tag{A 2.2}$$

where $\mathfrak{K}^{(1)}$ and $\mathfrak{K}^{(2)}$ are both small and do not contain oscillatory terms. It is the factor e^{iN} which might give rise to rapid oscillations in the derivatives of $\phi(\theta;\ N)$ and $\Lambda(u)$. The same procedure works for large N in an angle

$$0 \leqslant \arg N \leqslant 2\epsilon \quad (\text{i.e. } 0 \leqslant \arg u \leqslant \epsilon),$$

where e^{iN} remains bounded, provided that the kernels $\mathfrak{K}^{(1)}$ and $\mathfrak{K}^{(2)}$ remain small in this angle. This seems reasonable but needs detailed verification.

It is the angle $-2\epsilon \leqslant \arg N \leqslant 0$, however, in which we are really interested. Here the factor e^{iN} becomes exponentially large, and the method of iteration fails with the kernel (A 2.2). But we note that the term $e^{iN}\mathfrak{K}^{(2)}$ may be modified by subtracting a finite number m of wavefree potentials (Ursell 1953, p. 96, footnote). The resulting term is roughly of the form $(2m)!\,N^{-2m}e^{iN}\mathfrak{K}^{(2)}_{2m}(\theta,\alpha;\ N)$ where $\mathfrak{K}^{(2)}_{2m}$ is expected to be uniformly small. If now we take for m an integer near $\frac{1}{2}|N|$, then Stirling's formula shows that this part of the kernel is exponentially small, and an iterative solution can proceed very nearly as before. The kernels $\mathfrak{K}^{(1)}$ and $\mathfrak{K}^{(2)}$ must still be studied in detail, but it is not expected that this will present any great difficulty.

When $\phi(\theta;\ N)$ has been shown in this way to be non-oscillatory, it will then follow from the definition (see (3.6) above) that the behaviour of the force coefficient $\Lambda(u)$ is given by (A 2.1) in some positive angle $-\epsilon \leqslant \arg u \leqslant \epsilon$ and hence also that $\Lambda(u)$ and all its derivatives are non-oscillatory for large real u.

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